On Generating Systems of Ternary and Quaternary Linear Transformations.

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The following special cases of a general algebraic theorem admit in geometric form a proof so simple that the process which it postulates can be entirely visualized. Though we tacitly assume the reality of all points spoken of, this restriction is of no moment; for two points determine a line, three points a plane, whether the points be real or imaginary. The theorem now to be examined is fundamental in the Algebra of Linear Transformations. I shall exhibit proofs of the theorem only as particularized in the theories of ternary and quaternary forms, since beyond three dimensions our intuition of spacial figures ceases. A general proof will necessarily be of purely algebraic character.

In order to write as simply as possible the differential equations satisfied by every invariant of linear transformation of n homogeneous variables, one seeks a system of linear transformations of simplest type, such that by repetition and successive application of these elementary transformations the most general can be compounded. Such a system is termed a complete system of Generators of the n-ary group.

§1.—*Generators of the Ternary Group.*

Theorem: A complete generating system of ternary linear transformations is contained in the following five; three of which have the determinant equal to 1, *the other two a determinant different from* 1.

$$\text{I.}\quad x_1 = y_1 + \lambda \cdot y_2,\quad x_2 = y_2,\quad x_3 = y_3.$$

$$\text{II.}\quad x_1 = y_1,\quad x_2 = y_2 + \mu \cdot y_3,\quad x_3 = y_3.$$

$$\text{III.}\quad x_1 = y_1,\quad x_2 = y_2,\quad x_3 = y_3 + \nu \cdot y_1.$$

IV. $x_1 = y_1$,
$x_2 = \alpha . y_2$,
$x_3 = y_3$.

V. $x_1 = y_1$,
$x_2 = y_2$,
$x_3 = \beta . y_3$.

Interpreting $x_1 : x_2 : x_3$ and $y_1 : y_2 : y_3$ as point coordinates in a plane, I shall first give a geometric characterization of each of these five elementary transformations. The well-known method of determining completely by the use of two point-quadruples a general linear transformation will enable me, secondly, to restate in geometric terms the above theorem in the form of a problem. A simple process of solution suggests itself, whose verification finally proves the theorem.

By the substitutions IV and V each side and each vertex of the triangle of reference in the x-plane becomes the corresponding side or vertex of the triangle of reference in the y-plane. Calling the vertices A_1, A_2, A_3, the opposite sides a_1, a_2, a_3, I may say more briefly: Each of the six elements A_1, A_2, A_3, a_1, a_2, a_3 of the triangle of reference is transformed into itself. So is also, by IV, every line $ax_1 + cx_3 = 0$ passing through A_2, and by V, every line $ax_1 + bx_2 = 0$ passing through A_3. But every point not on the stationary line ($x_2 = 0$ in the one case, $x_3 = 0$ in the other) is transferred directly toward or directly from the fixed point A_2 or A_3. Hence I may call these transformations IV and V, loosely, *translations* of the x-plane.

Similarly the transformations I, II, III may be called for present purposes *rotations* of the plane. By the transformation I, for example, while the sides $x_2 = 0$ and $x_3 = 0$ become the sides $y_2 = 0$ and $y_3 = 0$ of the new triangle of reference, the side $y_1 = 0$ coincides, not with $x_1 = 0$, but with the line $x_1 - \lambda . x_2 = 0$. Every line through the vertex A_1 $ax_1 + bx_2 = 0$ becomes a line through the new vertex A'_1 $ay_1 + (a\lambda + b)y_2 = 0$. The transformation is described, with sufficient precision, as a *rotation of the side a_1 about the vertex A_3 as a center.* Fixing the attention upon the triangle of reference I see that—

By I, the side a_1 revolves about A_3 as a center.
" II, " " a_2 " " A_1 " " "
" III, " " a_3 " " A_2 " " "

The theorem gives as elementary operations, therefore, three rotations and two translations. By the aid of these special transformations there is to be produced the total effect of any given general collineation. What is the geometrical character of the latter?

Projective Geometry furnishes the theorem: "A collineation is completely determined when to any four points of the plane are assigned respectively the four points into which they are to be transformed; provided that no three points of either quadruple lie in a straight line." Taking as given points A_1, A_2, A_3, and P not lying on either a_1, a_2, or a_3, and assigning to them respectively four arbitrary points B_1, B_2, B_3, Q, I shall have proved the required theorem if I can always solve the following problem:

By the use of the three rotations and two translations above described, to transfer the three vertices A_1, A_2, A_3 of the triangle of reference, and a fourth point P not lying upon any side of that triangle, to four arbitrary points B_1, B_2, B_3, Q respectively, where no three of the latter four points are collinear.

The solution is effected as follows. If the three vertices A_1, A_2, A_3 were already brought to occupy the positions B_1, B_2, B_3, they could remain fixed while the two translations should be applied to transfer the fourth point, P, to the position Q. Now the three rotations suffice for transferring the vertices A_1, A_2, A_3 to the desired points; for by their aid it is possible to transfer any one, *e. g.* A_1, to an arbitrary point B_1 and leave the other two, A_2 and A_3, at their initial positions. To see the truth of this assertion, suppose that B_1 is taken upon the a_3. A rotation III withdraws the side a_3 from the point B_1, and transfers the vertex A_1 along the side a_2 for an arbitrary distance to some point A'_1. If B_1 were taken not lying upon a_3, A_1 may be said to coincide with A'_1. Next

By II, let a_2 be brought to contain B_1; incidentally the vertex A_3 moves upon a_1 to a position A'_3;

By III, let a_3 be brought to contain B_1; this completes the transfer of A'_1 to B_1; finally—

By II, let a_2 be revolved about B_1 till its intersection with a_1 returns from A'_3 to its initial position A_3.

Thus by at most four rotations, III, II, III, II, A_1 has been transferred to B_1, while A_2 and A_3 are unchanged. If P has been brought to P', then next in order—

By I, III, I, III, the points B_1, A_2, A_3, P' may become B_1, B_2, A_3, P'';
" II, I, II, I, " " B_1, B_2, A_3, P'' " " B_1, B_2, B_3, P'''.

If now the lines $\overline{B_2P''}$ and $\overline{B_3Q}$ intersect, or if the lines $\overline{B_2Q}$ and $\overline{B_3P'''}$ are not parallel, two translations will complete the required transfer; otherwise three will be necessary. Neglecting the latter possibility, since it adds no difficulty to the problem, suppose $\overline{B_2P''}$ and $\overline{B_3Q}$ to intersect in a point P''''.

By IV, let P'' be brought to the position P'''';
" V, " P'''' " " " Q.

These translations leave B_1, B_2, B_3 fixed, and bring the point whose original position was P to the terminal position Q. The above problem is therefore solved, and the theorem stated at the outset is proved.

Logically more symmetrical, but practically less brief would be the solution, if made to depend on the lemma: *By using not more than seven operations chosen from the three rotations and two translations, it is possible to transfer any one of the four points A_1, A_2, A_3, P, to the corresponding point B_1, B_2, B_3 or Q in such a way that the terminal positions of the other three shall coincide with their initial positions.* The proof is sufficiently obvious from the foregoing. An interesting exceptional case would arise, for example, if the points B_1, B_2, B_3 were situated upon the sides a_1, a_2, a_3 respectively, and the fourth point Q lay upon any one side a_1, a_2, or a_3 of the triangle of reference.

§2.—*Generators of the Quaternary Group.*

For the group of linear transformations of four homogeneous coordinates $x_1 : x_2 : x_3 : x_4$ in three-dimensional space an obvious extension of the theorem of §1 is the following:

Theorem: A complete generating system of quaternary linear transformations is contained in the following seven, each of which contains a single variable parameter.

I. $x_1 = y_1 + \lambda_1 y_2$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_4$.

II. $x_1 = y_1$, $x_2 = y_2 + \lambda_2 y_3$, $x_3 = y_3$, $x_3 = y_4$.

III. $x_1 = y_1,$ $x_2 = y_2,$ $x_3 = y_3 + \lambda_3 y_4,$ $x_4 = y_4.$		IV. $x_1 = y_1,$ $x_2 = y_2,$ $x_3 = y_3,$ $x_4 = y_4 + \lambda_4 y_1.$
V. $x_1 = y_1,$ $x_2 = \alpha . y_2,$ $x_3 = y_3,$ $x_4 = y_4.$	VI. $x_1 = y_1,$ $x_2 = y_2,$ $x_3 = \beta . y_3,$ $x_4 = y_4.$	VII. $x_1 = y_1,$ $x_2 = y_2,$ $x_3 = y_3,$ $x_4 = \gamma . y_4.$

The order and method of proof employed in the preceding case can be followed again here. Any particular value of the arbitrary parameter in each of these seven elementary transformations shall be regarded as the terminal value of a quantity which has varied continuously from an initial value zero. Each such finite variation of a parameter can be represented by a system of motions of finite magnitude in three-dimensional space. In the tetraedron of reference let each face: $x_i = 0$ be named the face a_i; let each vertex opposite a face a_i be called A_i, and each edge $\overline{A_i A_\kappa}$ be called $\alpha_{i\kappa}$. The above seven transformations may then be termed respectively *axial rotations* (I, II, III, IV) or *translations* (V, VI, VII). The effects which are relevant to the present purpose may be noted briefly, as follows: Each axial rotation leaves *in situ* three faces, three vertices, and four edges, of the tetraedron of reference; but causes to revolve about one edge as an axis every plane, save one, containing that edge,—among others the fourth face, containing the fourth vertex, of the tetraedron of reference. More particularly—

Rotation	I,	about	an axis	α_{34},	shifts	the face	a_1	and vertex	A_2,
"	II,	"	" "	α_{41},	"	" "	a_2	" "	A_3,
"	III,	"	" "	α_{12},	"	" "	a_3	" "	A_4,
"	IV,	"	" "	α_{23},	"	" "	a_4	" "	A_1.

Each translation leaves *in situ* every vertex of the tetraedron of reference, but transfers directly toward or from some one vertex every point not lying in the opposite face of the tetraedron. For the translations V, VI, VII, the central vertices are respectively A_2, A_3, A_4.

Since a collineation is completely determined when it is required to transform five given points into arbitrary points respectively, no four points in either quintuple being co-planar, the proof of the above theorem will be contained in the solution of the problem:

By means of operations chosen from the four sorts of rotation, and the three sorts of translation just described, to transfer the four vertices A_1, A_2, A_3, A_4 *of the tetraedron of reference and a fifth point* P *not lying in any face thereof, to five arbitrary positions* B_1, B_2, B_3, B_4 and Q *respectively, no four of which lie in the same plane.*

The solution will correspond, step for step, to that adopted in the ternary case. Each point A_1, A_2, A_3, or A_4 in succession can be transferred by five, six, or seven *rotations* to the corresponding terminal position while the remaining three vertices remain undisturbed or else are restored to their original positions. If B_1, for example, is not in either face a_3 or a_4, then by the following series of rotations:

II, III, IV, III, II, the points

A_1, A_2, A_3, A_4, P can be brought to the

positions B_1, A_2, A_3, A_4, P'. Similarly three other sets of five or more rotations can be applied to transfer these five points to the positions:

$$B_1,\ B_2,\ B_3,\ B_4,\ P^{(\mathrm{IV})}.$$

It remains only to bring the point $P^{(\mathrm{IV})}$ to the position Q by means of translations. If either

the plane $(A_2, A_3, P^{(\mathrm{IV})})$ is not parallel to $\overline{A_4Q}$,
or " " $(A_3, A_4, P^{(\mathrm{IV})})$ " " " " $\overline{A_2Q}$,
" " " $(A_4, A_2, P^{(\mathrm{IV})})$ " " " " $\overline{A_3Q}$,

then three translations suffice to effect the object; otherwise, four. Thus we have, for the general case, a complete solution of the problem, which is tantamount to a proof of the theorem. In the most highly specialized case, four arbitrary rotations of the four sorts and any one arbitrary translation would reduce the problem to one of general character. Hence for all cases the theorem is established.

§3.—*Algebraical Determination of the Generators of a given Ternary Linear Transformation.*

When the nine coefficients of a ternary transformation:

$$\begin{aligned}\rho x_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3\\ \rho x_2 &= a_{21}y_1 = a_{22}y_2 + a_{23}y_3\\ \rho x_3 &= a_{31}y_1 = a_{32}y_2 + a_{33}y_3\end{aligned}$$

are given, and it is desired to find the numerical values of the parameters in its generators, the scheme of §1 gives a most convenient algorithm for the purpose. First we must determine the coordinates of the positions into which the four points—

$$\begin{cases} A_1, \text{ whose coordinates are } 1:0:0 \\ A_2, \quad \text{"} \qquad \text{"} \qquad \text{"} \quad 0:1:0 \\ A_3, \quad \text{"} \qquad \text{"} \qquad \text{"} \quad 0:0:1 \\ P_1 \quad \text{"} \qquad \text{"} \qquad \text{"} \quad p_1, p_2, p \end{cases}$$

are to be transferred. These are readily seen. Calling the determinant of the substitution D,—

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and its minors } D_{i\kappa},\text{—}$$

$D_{ik} = \dfrac{\partial D}{\partial a_{ik}}$, we have as coordinates $(b_{i1} : b_{i2} : b_{i3})$ of each point B_i, $(i = 1, 2, 3)$, the following:

$$\left.\begin{cases} \sigma . b_{i1} = D_{i1} \\ \sigma . b_{i2} = D_{i2} \\ \sigma . b_{i3} = D_{i3} \end{cases}\right\} \begin{matrix} (i = 1, 2, 3), \\ \sigma = \dfrac{D}{\rho}. \end{matrix}$$

P is transformed into Q with the coordinates:

$$\left.\begin{cases} \sigma . q_1 = p_1 . D_{11} + p_2 . D_{21} + p_3 . D_{31} \\ \sigma . q_2 = p_1 . D_{12} + p_2 . D_{22} + p_3 . D_{32} \\ \sigma . q_3 = p_1 . D_{13} + p_2 . D_{23} + p_3 . D_{33} \end{cases}\right\}.$$

The algebraic equivalent of transferring either one of the four points to the required terminal position is the determination of suitable values for two parameters. There are therefore in all eight parameters to be fixed—exactly the number of coefficients, barring the factor of proportionality, that occur in the given linear transformation.

For effecting the transfer of A_1 to B_1 while A_2 and A_3 remain *in situ*, the generators have the form,—

$$\left\{\begin{array}{l|l|l} x_1 = x_1' & x_1' = x_1'' & x_1'' = x_1''' \\ x_2 = x_2' + \mu_1 x_3' & x_2' = x_2'' & x_2'' = x_2''' - \mu_1 x_3''' \\ x_3 = x_3' & x_3' = x_3'' + \nu_1 x_1'' & x_3'' = x_3''' \end{array}\right.$$

The composition of these gives the substitution,—

$$\begin{cases} x_1 = x_1''' \\ x_2 = x_2''' + \mu_1\nu_1 x_1''' \\ x_3 = x_3''' + \nu_1 x_1''' \end{cases}$$

Inserting the coordinates of A_1 and B_1, we have to determine ν_1, μ_1, the formulæ:

$$\begin{cases} \sigma . 1 = D_{11} \\ \sigma . 0 = D_{12} + \mu_1\nu_1 D_{11} \\ \sigma . 0 = D_{13} + \nu_1 D_{11} \end{cases} \quad \Bigg| \quad \therefore \begin{cases} \nu_1 = -\dfrac{D_{13}}{D_{11}} \\ \mu_1 = +\dfrac{D_{12}}{D_{13}} \end{cases}$$

Introducing these values of μ_1, ν_1, we find the coordinates of P',—

$$\begin{cases} p_1' = p_1 \\ p_2' = p_2 + \dfrac{D_{12}}{D_{11}} \cdot p_1 \\ p_3' = p_3 + \dfrac{D_{13}}{D_{11}} \cdot p_1 \end{cases}$$

In like manner, corresponding to the removal of A_2 and A_3 respectively, the two pairs of parameters ν_2, λ_2; λ_3, μ_3 would be found, and thence successively the coordinates of P'' and P'''. To transform the latter point into Q, there are to be determined finally the values of α, β from the formulæ (see IV and V, §1),—

$$\begin{cases} \sigma . p_1''' = \sigma . q_1 \quad = p . D_{11} \quad + p_2 . D_{21} + p_3 . D_{31} \\ \sigma . p_2''' = \alpha . \sigma . q_2 = \alpha . (p_1 . D_{12} + p_2 . D_{22} + p_3 . D_{32}) \\ \sigma . p_3''' = \beta . \sigma . q_3 = \beta . (p_1 . D_{13} + p_2 . D_{23} + p_3 . D_{33}) \end{cases}$$

Thus all eight parametric values are found from linear equations. The solution of these equations can become impossible only by the vanishing of one or more first minors of the determinant D. To the treatment of such exceptional cases the geometric considerations of §1 furnish still a perfect guide, requiring the insertion of additional generators, with arbitrary (restricted) parameters, in the system of eleven which suffices for the general case.

The analogous algorithm for factoring any given quaternary linear transformation into generators of the types here prescribed is sufficiently obvious, arising out of the scheme of §2 just as that given above for ternary forms arose from the scheme of §1. We see readily that it will fix the values of 15 parameters involved in 23 or more generators; and that if for a particular trans-

formation more than 23 generators are requisite, all the parameters beyond 15 will be arbitrary. Further, it is worthy of remark that this algorithm, once found, is readily extended to determining similarly defined generators of an n-ary linear transformation. Hence there would be no difficulty in inferring that n *rotations* and $n-1$ *translations* must comprise a complete system of generators for the linear transformations of n homogeneous variables.

CLARK UNIVERSITY, WORCESTER, MASS., *March* 22, 1892.